Induced separation dimension

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Abstract. A linear ordering of the vertices of a graph G *separates* two edges of G if both the endpoints of one precede both the endpoints of the other in the order. We call two edges $\{a, b\}$ and $\{c, d\}$ of G *strongly independent* if the set of endpoints $\{a, b, c, d\}$ induces a $2K_2$ in G. The *induced separation dimension* of a graph G is the smallest cardinality of a family $\mathcal{L}$ of linear orders of $V(G)$ such that every pair of strongly independent edges in G are separated in at least one of the linear orders in $\mathcal{L}$. For each $k \in \mathbb{N}$, the family of graphs with induced separation dimension at most k is denoted by ISD(k).
In this article, we initiate a study of this new dimensional parameter. The class ISD(1) or, equivalently, the family of graphs which can be embedded on a line so that every pair of strongly independent edges are disjoint line segments, is already an interesting case. On the positive side, we give characterizations for chordal graphs in ISD(1) which immediately lead to a polynomial time algorithm which determines the induced separation dimension of chordal graphs. On the negative side, we show that the recognition problem for ISD(1) is NP-complete for general graphs. We then briefly study ISD(2) and show that it contains many important graph classes like outerplanar graphs, chordal graphs, circular arc graphs and polygon-circle graphs. Finally, we describe two techniques to construct graphs with large induced separation dimension. The first one is used to show that the maximum induced separation dimension of a graph on n vertices is $\Theta(\lg n)$ and the second one is used to construct AT-free graphs with arbitrarily large induced separation dimension.

1 Introduction

Vertex orderings which meet certain local conditions have turned out to be a very useful tool in the study of graphs. Perfect elimination orderings of a chordal graph is perhaps the most striking example. Graph families like comparability graphs, interval graphs, unit interval graphs, strongly chordal graphs and threshold graphs can be characterized based on the existence of a vertex ordering with a certain simple property [4, 8]. Such orderings are useful not just in providing structural insights into the family, but also in designing efficient algorithms on those families for problems which are NP-hard on general graphs. In addition, some of these algorithms can be extended to a larger family formed by working with a small family of vertex orderings rather than a single one. Such extensions have resulted in the introduction of many "dimensional" parameters on graphs like boxicity [18], cubicity [18], threshold dimension [7], hypergraph dimension [10], separation dimension [2], etc. and efficient algorithms on families in which one of these dimensions is bounded.

In this article, we use vertex orderings to define a graph parameter, which we call "induced separation dimension", and show that several interesting classes of graphs have a small induced separation dimension.

Let σ be a linear order on the elements of a set U. For two disjoint subsets A and B of U, we say $A \prec_\sigma B$ when every element of A precedes every element of B in σ, i.e., $a \prec_\sigma b,\ \forall(a,b) \in A \times B$. We say that σ *separates* A and B if either $A \prec_\sigma B$ or $B \prec_\sigma A$.

Definition 1 (Induced separation dimension). Two edges $\{a,b\}$ and $\{c,d\}$ of a graph G are called *strongly independent* if $G[\{a,b,c,d\}]$, the subgraph of G induced on vertices $\{a,b,c,d\}$, is isomorphic to $2K_2$, the disjoint union of two edges. A family $\mathcal{L}$ of linear orders of $V(G)$ is called *weakly separating* if every pair of strongly independent edges in G is separated in at least one order in $\mathcal{L}$. The smallest cardinality of a weakly separating family of linear orders for G is called the *induced separation dimension* of G and is denoted by $\mathrm{isd}(G)$. For each $k \in \mathbb{N}$, the family of graphs with induced separation dimension at most k is denoted by $\mathrm{ISD}(k)$.

For example, one may easily check that a complete graph, a chordless path on at least 5 vertices and a chordless cycle on at least 6 vertices have induced separation dimension, respectively, 0, 1 and 2. Indeed, a graph G has induced separation dimension 0 if and only if G is $2K_2$-free or, equivalently, if the complement graph $\overline{G}$ is C_4-free. Hence, $\mathrm{ISD}(0) = \{G : G \text{ is } 2K_2\text{-free}\} = \{G : \overline{G} \text{ is } C_4\text{-free}\}$. The family of $2K_2$-free graphs have received considerable attention in literature, resulting in many structural, algorithmic and extremal results [6, 16, 5]. The left endpoint order of an interval representation of an interval graph separates every pair of strongly independent edges. Hence, interval graphs belong to $\mathrm{ISD}(1)$. Every pair of strongly independent edges in a (rooted) tree is separated either in the DFS pre-order or in the DFS post-order traversal. Thus, trees belong to $\mathrm{ISD}(2)$.

Relation to separation dimension. The cardinality of a smallest family $\mathcal{L}$ of linear orders on the vertices of a graph G such that every pair of non-incident edges (two edges with no common endpoints) is separated in at least one of the linear orders in $\mathcal{L}$ is called the *separation dimension* of G [2]. There has been a detailed recent study on the separation dimension of graphs and hypergraphs [1–3]. It follows by definition that the induced separation dimension of a graph is at most its separation dimension. In particular, the induced separation dimension of an n-vertex graph is at most $O(\lg n)$ [3].

But, what we find more interesting is the difference between the two notions. One of the main sources of this difference is that, while separation dimension is a monotone parameter (adding edges cannot decrease the separation dimension of a graph), induced separation dimension is not. Thus, dense graphs, even if highly structured, tend to have large separation dimension. On the other hand, induced separation dimension of some dense but structured graph families is very low. For instance, while separation dimension of cocomparability graphs and chordal graphs is unbounded, their induced separation dimension, as we establish here, is bounded above by 1 and 2 respectively. Their difference is also highlighted by the fact that while the family of graphs with separation dimension 1 has a complete characterization which leads to an easy linear-time recognition algorithm [3], we show here that it is NP-complete to decide whether a graph belongs to ISD(1).

1.1 Results and organization

We begin by showing that a weakly separating family of linear orders for a graph G corresponds closely with a special family of acyclic orientations of the complement graph $\overline{G}$ (Section 2). This characterization is later used to derive both upper and lower bounds on induced separation dimension and also to establish an NP-hardness result.

In Section 3, we focus on the graph class ISD(1), i.e., graphs with a single vertex ordering that separates every pair of strongly independent edges. The characterization mentioned above helps us conclude that all cocomparability graphs belong to ISD(1). The same characterization is also used to establish NP-hardness of the recognition problem for ISD(1). We then describe a forbidden configuration for graphs in ISD(1), namely, an asteroidal triple of edges (ATE) and show that a chordal graph belongs to ISD(1) if and only if it is ATE-free. We also note that a tree belongs to ISD(1) if and only if it is a caterpillar with toes.

In Section 4, we go one step further and briefly study the graph class ISD(2). The main result here is that ISD(2) contains the class of interval filament graphs. Since the class of interval filament graphs contains many important graph classes like chordal graphs, circular arc graphs and polygon-circle graphs, we conclude that all of them belong to ISD(2). Since chordal graphs belong to ISD(2) and the characterization of chordal graphs in ISD(1) as ATE-free graphs is testable in polynomial time, we get a poly-time algorithm to determine the induced separation dimension of chordal graphs. From the literature on separation dimension, we know that outerplanar graphs belong to ISD(2) and planar graphs belong to ISD(3) [3]. We do not yet know whether planar graphs belong to ISD(2).

Finally, in Section 5, we describe two techniques to construct graphs with large induced separation dimension. Using the first one, we construct n-vertex graphs with induced separation dimension at least $\lg n$, showing that the upper bound of $O(\lg n)$ which follows from the relation to separation dimension is tight up to a constant factor. The second construction is used to show that the family of AT-free graphs have unbounded induced separation dimension, in stark contrast to its subfamily of cocomparability graphs.

1.2 Notations and definitions

All graphs we study here are finite and simple. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$ respectively. For a graph G and $S \subset V(G)$, the subgraph of G induced on S is denoted by $G[S]$. The complement graph of G is denoted by $\overline{G}$. A graph is called H-free if it has no induced subgraph isomorphic to H. For a vertex v of G, $N(v)$ denotes the set of neighbors of v and $N[v] = N(v) \cup \{v\}$.

The complete graph and the chordless cycle on n vertices are denoted, respectively, by K_n and C_n. The vertex disjoint union of k different copies of a graph is denoted by kG. In particular $2K_2$ denotes two strongly independent edges.

A *cocomparability* graph is an undirected graph that connects pairs of elements that are *not* comparable to each other in a partial order, i.e., the complement of a comparability (transitively orientable) graph. A graph is called *chordal* if it has no induced cycles of size strictly greater than 3. An *interval graph* is an intersection graph of intervals on the real line, and a *unit interval graph* is an intersection graph of unit length intervals on the real line. An independent triple of vertices x, y, z in a graph G is an *asteroidal triple* (*AT*) if, between every pair of vertices in the triple, there is a path that does not contain any neighbor of the third. A graph without asteroidal triples is called an *asteroidal triple-free (AT-free) graph.* A graph is *outerplanar* if it has a crossing-free embedding in the plane such that all vertices are on the same face. A *caterpillar* is a tree with a dominating path, and a *caterpillar with toes* is a tree with a 2-step dominating path. A 2-*step dominating path* in a graph G is a path P such that every vertex of G is at distance at most 2 from P.

2 Linear orders and orientations of the complement

We start by giving a graph invariant that is equal to the induced separation dimension of the complement graph. This equivalent view will be useful in some of the proofs to come later.

Definition 2 (C_4-transitive orientations). An acyclic orientation of an undirected simple graph G is an assignment of directions to each edge of G so that no directed cycles are formed. A family $\mathcal{O}$ of acyclic orientations of G is called C_4-*transitive* on G if every induced C_4 in G is oriented transitively in at least one orientation in $\mathcal{O}$. The minimum cardinality of a C_4-transitive family of acyclic orientations of G is denoted by $\eta(G)$.

Theorem 3. *For every undirected simple graph G,*

$$\mathrm{isd}(G) = \eta(\overline{G}).$$

Proof. Let $\mathcal{L}$ be a family of linear orders that is weakly separating for G. For every linear order $\sigma \in \mathcal{L}$ we define an orientation O_σ of $\overline{G}$ as follows. An edge

$\{u, v\}$ of $\overline{G}$ where $u \prec_\sigma v$ is oriented from u to v (denoted $\overrightarrow{uv}$). This orientation of $\overline{G}$ is obviously acyclic. We claim that the family of acyclic orientations $\{O_\sigma : \sigma \in \mathcal{L}\}$ is C_4-transitive on $\overline{G}$. Let (a, b, c, d) be an induced C_4 in $\overline{G}$. Then the pair of edges ac and bd forms an induced $2K_2$ in G. Let $\sigma \in \mathcal{L}$ be the total order which separates the edges ac and bd of G. That is, we have either $\{a, c\} \prec_\sigma \{b, d\}$ or $\{b, d\} \prec_\sigma \{a, c\}$. In both cases, it is easy to check that O_σ is transitive on the cycle (a, b, c, d).

In the other direction, given a family $\mathcal{O}$ of acyclic orientations that is C_4-transitive on $\overline{G}$, we construct a family of total orders $\{\prec_O : O \in \mathcal{O}\}$ on $V(G)$, where for each $O \in \mathcal{O}$, the total order $\prec_O$ is a linear extension of the transitive closure of O. We claim that $\{\prec_O : O \in \mathcal{O}\}$ is weakly separating for G. Let the pair of edges ab and cd be an induced $2K_2$ in G. Then (a, c, b, d) is an induced C_4 in $\overline{G}$. Let $O \in \mathcal{O}$ be the orientation of $\overline{G}$ which is transitive on (a, c, b, d). There are only two possible transitive orientations for this cycle, namely $\{\overrightarrow{ac}, \overrightarrow{ad}, \overrightarrow{bc}, \overrightarrow{bd}\}$ and the orientation obtained by reversing all the directions in the first one. It is easy to check that $\{a, b\} \prec_O \{c, d\}$ in the first case and $\{c, d\} \prec_O \{a, b\}$ in the second case. □

3 The graph class ISD(1)

The following corollary is a restatement of Theorem 3 for ISD(1) and the next one is then immediate.

Corollary 4. *A graph G belongs to* ISD(1) *if and only if there exists an acyclic orientation of $\overline{G}$ which is transitive on every induced 4-cycle of $\overline{G}$.*

Corollary 5. *The family of cocomparability graphs is contained in* ISD(1)*.*

Remark. The path on 5-vertices P_5 is an interval graph and has a pair of strongly independent edges. Hence, interval graphs and thereby cocomparability graphs are not contained in ISD(0).

Next we use Corollary 4 to show that the recognition problem for ISD(1) is NP-hard. We do this by reducing the 2-coloring problem on 3-uniform hypergraphs to the problem of deciding whether $\eta(G) \leq 1$ for a graph G.

A 3-*uniform hypergraph* H over a set of vertices V is a collection of 3-element subsets of V, called hyperedges. A *proper coloring* of H is a coloring of V so that every hyperedge in H contains vertices of at least two different colors. A hypergraph is called 2-*colorable* if it can be properly colored using 2 colors. It is a result of Lovász from 1973 that testing 2-colorability of 3-uniform hypergraphs is NP-hard [15].

Theorem 6. Problem 1 *below is polynomial-time reducible to* Problem 2*.*

Problem 1. *Given a 3-uniform hypergraph H, decide whether H is 2-colorable.*
Problem 2. *Given a graph G, decide whether $\eta(G) \leq 1$.*

Proof. Let H contain n vertices $v_1, \ldots, v_n$ and m hyperedges $e_1, \ldots, e_m$. Let L be a bipartite graph on $6m$ vertices with color classes $A = \{a_1, \ldots, a_{3m}\}$ and $B = \{b_1, \ldots, b_{3m}\}$. Vertices a_i and b_j are adjacent in L if and only if $|i - j| \leq 1$. (L is a $3m$-ladder graph). For each $i \in [3m-1]$, $(a_i, b_i, a_{i+1}, b_{i+1})$ is an induced C_4 in L and these are all the induced C_4's in L. There are only two orientations of L which are transitive on every induced C_4; one which orients every edge from A-side to B-side and the other which orients every edge from B-side to A-side.

To construct G, we first associate a different copy $L(v)$ of the ladder L for each vertex v of H. For each hyperedge $e_l = \{v_i, v_j, v_k\}$, $i < j < k$, we glue together the three ladders $L(v_i)$, $L(v_j)$ and $L(v_k)$ at their $3l$-th level as follows: the vertex b_{3l} of $L(v_i)$ is identified with the vertex a_{3l} of $L(v_j)$; b_{3l} of $L(v_j)$ with a_{3l} of $L(v_k)$; and b_{3l} of $L(v_k)$ with a_{3k} of $L(v_k)$; forming a 3-cycle. These identifications do not create any new induced 4-cycles since we have chosen to skip 3 levels of the ladder after the modification for each hyperedge. This completes the construction of the graph G given the hypergraph H and it is clearly polynomial time. We complete the proof by showing that $\eta(G) \leq 1$ if and only if H is 2-colorable.

Suppose that H is 2-colorable and let $\phi : V(H) \to \{0, 1\}$ be a proper coloring of H. Orient the edges of G as follows. If $\phi(v) = 0$, orient every edge of $L(v)$ in G from A-side to B-side and if $\phi(v) = 1$, orient every edge of $L(v)$ from B-side to A-side. Since all the induced 4-cycles in G are subgraphs of the constituent ladders, they are all oriented transitively. All the 3-cycles formed by the hyperedges are oriented acyclically since each of them contains two vertices of different colors. For every longer cycle C (length 4 or more), at least two consecutive edges of C belong to the same ladder and hence C is oriented acyclically. Thus, the above orientation of G is transitive on every induced C_4 and acyclic. Thus $\eta(G) \leq 1$.

In the other direction, suppose $\eta(G) \leq 1$ and let O be an acyclic orientation of G that is transitive on every induced C_4 in G. As noted above, there are only two possible orientations for each ladder that is transitive on every induced C_4. Define a coloring $\phi : V(H) \to \{0, 1\}$ based on O as follows: $\phi(v) = 0$ if the edges of $L(v)$ in G are oriented from A-side to B-side and $\phi(v) = 1$ otherwise, i.e., if every edge of $L(v)$ is oriented from B-side to A-side. Since O is an acyclic orientation, the 3-cycle corresponding to each hyperedge of H is oriented acyclically in O. That is, every hyperedge contains vertices of both colors under ϕ. Thus, ϕ is a proper 2-coloring of H. □

Since Problem 1 defined in Theorem 6 is NP-hard [15], so is Problem 2. Moreover, Problem 2 is in NP since the number of induced 4-cycles in a graph is polynomial in the number of vertices. Hence, by Corollary 4, we conclude the following.

Corollary 7. *The recognition problem for* ISD(1) *is NP-complete.*

Next, we give a configuration that is forbidden for graphs in ISD(1). This will turn out to be useful in characterizing trees and chordal graphs in ISD(1). The closed neighborhood of an edge $\{u, v\}$ in a graph G is the set $N[u] \cup N[v]$.

Definition 8 (ATE-free graph). An *asteroidal triple of edges* (*ATE*) in a graph G is a collection of three edges in G such that, between every pair of them, there exists a path in G which does not contain any vertex in the closed neighborhood of the third edge. A graph without an ATE is called *ATE-free.*

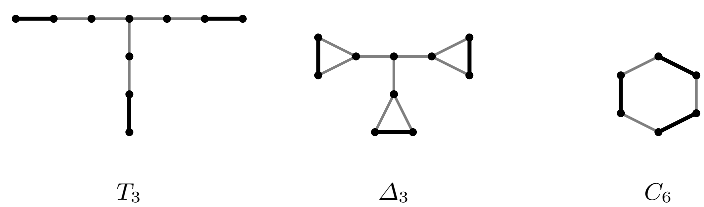

Fig. 1. Examples of graphs with an asteroidal triple of edges. The three edges which form an asteroidal triple are drawn with thicker lines.

Some examples of graphs with an ATE are depicted in Fig. 1. Any ATE-free graph is thus T_3-free, Δ_3-free, C_6-free and so on.

Remark. Note that the three edges of an ATE themselves need not be pairwise strongly independent, as illustrated by the cycle C_6. Nevertheless, one can verify that all AT-free graphs are ATE-free.

Theorem 9. *All graphs in* ISD(1) *are ATE-free.*

Proof. Let $G \in \text{ISD}(1)$ and $\prec$ be a single linear order that separates all the strongly independent edges in G. Suppose, for the sake of contradiction, that G contains an ATE. Let aa', bb', and cc' be the three edges forming an ATE in G. Let P_a be the path between bb' and cc' which does not contain any vertex in the closed neighborhood of aa'. P_b and P_c are defined similarly. It is clear that $\prec$ separates the edge xx' from the set $V(P_x)$, for each $x \in \{a, b, c\}$. This demands that no edge of the ATE is completely sandwiched between the endpoints of another. Next we show that one of the above two conditions is violated by $\prec$. This contradiction shall prove the theorem.

Let $S = \{a, a', b, b', c, c'\}$. We can assume, after relabelling if necessary, that $a \prec a'$, $b \prec b'$, $c \prec c'$ and $a \prec b \prec c$. So a is the first vertex of S in $\prec$. The next vertex of S in $\prec$ is not a', since in that case bb' is not separated from $V(P_b)$. Hence, the second vertex from S in $\prec$ is b. The third vertex is not a' for the same reason. Neither is it b' since, in that case bb' is sandwiched between a and a'. Hence, the third vertex is c. The fourth vertex is a' since otherwise either bb' or cc' edge will be sandwiched between a and a'. The fifth vertex has to be b' since otherwise cc' will be sandwiched between b and b'. The sixth vertex is c' by exhaustion. Thus, $a \prec b \prec c \prec a' \prec b' \prec c'$. But in this case, $V(P_b)$ is not separated from bb'. □

The converse of Theorem 9 is not true in general. We show later that the family of AT-free graphs and thereby the family of ATE-free graphs is not contained in ISD(k) for any constant k. Nevertheless, we show next that the converse of Theorem 9 is true for chordal graphs, i.e., ATE-free chordal graphs belong to ISD(1). We need to define a new notion to streamline the characterization.

Definition 10 (FAT-free graph). A vertex v in a graph G is called simplicial if $N(v)$ induces a clique in G. We call v *lonely* if v is simplicial but no neighbor of v is simplicial. An asteroidal triple A in G is called *fat* if none of the three vertices in A is lonely. The graph G is called *FAT-free* if it contains no fat asteroidal triples.

Hence, every asteroidal triple of vertices in a FAT-free graph has a simplicial vertex with no simplicial neighbor. We also need one observation regarding chordal graphs with an AT.

Observation 11. *If G is a chordal graph with an asteroidal triple, then G contains an independent set of three simplicial vertices.*

This observation can be verified by looking at a representation of G as an intersection graph of subtrees of a host tree T with the additional property that each node of T corresponds to a different maximal clique in G [12, Theorem 4.8]. Hence, each leaf of T is a subtree in the intersection model. These subtrees correspond to an independent set of simplicial vertices in G. Since G has an AT, the host tree T is not a path and therefore has at least 3 leaves.

Recalling that a *caterpillar* is a tree with a dominating path, we now state and prove a characterization for chordal graphs in ISD(1).

Theorem 12. *For a chordal graph G, the following are equivalent:*

(i) $G \in$ ISD(1).
(ii) *G is ATE-free.*
(iii) *G is FAT-free.*
(iv) *G is an intersection graph of distinct subtrees of a caterpillar.*

The proof is moved to the full version.

Remark. The requirement that the subtrees are *distinct* is essential in Condition (iv) above. The family of graphs which have a representation as the intersection graph of (not necessarily distinct) subtrees of a caterpillar are called *catval* graphs. The graph Δ_3 depicted in Fig. 1 is a catval graph but it has an ATE and therefore cannot be represented as an intersection graph of *distinct* subtrees of a caterpillar. Catval graphs were introduced by Jan Arne Telle in [19] and further studied by Habib, Paul and Telle in [14]. The tolerance version was studied by Eaton and Faubert in [9]. The proof that (iii) $\implies$ (iv) in the above theorem mimics a similar proof in [9].

We conclude this section by specializing the above characterization for trees in ISD(1). Recall that a caterpillar with toes is a tree with a 2-step dominating path.

Theorem 13. *For a tree T the following are equivalent:*

(i) $T \in \mathrm{ISD}(1)$.
(ii) T *is ATE-free.*
(iii) T *is T_3-free.*
(iv) T *is a caterpillar with toes.*

Proof. Theorem 12 establishes the equivalence of (i) and (ii). (ii) $\implies$ (iii) since T_3 contains an ATE. Any longest path in a T_3-free tree is 2-step dominating [13] and thus, (iii) $\implies$ (iv). One can verify easily that (iv) $\implies$ (ii) by a case analysis. □

Remark. More characterizations of caterpillars with toes can be found in [13, Theorem 3.7].

4 The graph class ISD(2)

Since outerplanar graphs have separation dimension at most 2 [3], they also have induced separation dimension at most 2. This bound is tight since C_6 is outerplanar and $\mathrm{isd}(C_6) > 1$. In this section, we show that interval filament graphs, a class introduced by Gavril [11], belongs to ISD(2). Interval filament graphs contain many well known graph classes like chordal graphs, circular-arc graphs (intersection graphs of arcs on a circle), polygon-circle graphs (intersection graphs of a convex polygons inscribed in a circle), etc. Thus, all of the above families belong to ISD(2). Since $\mathrm{isd}(C_6) = 2$, and C_6 is both a circular-arc graph and a polygon-circle graph, both these classes are not contained in ISD(1).

Definition 14 (Interval filament graph [11]). Let $\mathcal{I}$ be a collection of intervals on a horizontal line L embedded in a plane. In the half-plane above L, construct corresponding to each interval $I \in \mathcal{I}$ a curve f_I connecting the two endpoints of I such that f_I remains within the limits of I. The curve f_I is called an *interval filament* above I. A graph is an *interval filament graph* if it has an intersection model consisting of interval filaments.

Theorem 15. *The family of interval filament graphs are contained in* ISD(2)*.*

Proof. Let G be an interval filament graph and $(\mathcal{I}, \mathcal{F})$ be an interval filament intersection model of G. That is, each vertex v of G has an associated interval $I_v \in \mathcal{I}$ on a horizontal line L, and an interval filament $f_v \in \mathcal{F}$ above I_v such that G is the intersection graph of $\mathcal{F}$. Also define $l(v)$ and $r(v)$ to be, respectively, the left and right endpoints of I_v.

Let $\prec_l$ and $\prec_r$ be two linear orders on $V(G)$ such that $l(u) < l(v) \implies u \prec_l v$ and $r(u) < r(v) \implies u \prec_r v$. We argue that any pair of strongly independent edges ab and cd are separated in one of the two permutations above. If two vertices u and v are non-adjacent in G, then the corresponding intervals I_u and I_v are either disjoint or one is contained in the other. Without loss of generality, let a be the vertex with the leftmost left endpoint among $\{a, b, c, d\}$ and c be the vertex with the leftmost left endpoint among $\{c, d\}$. If ab is not separated from cd in $\prec_l$, then $l(a) < l(c) < l(b)$. In this case, since ab is an edge of G, $I_a \cap I_b \neq \emptyset$, hence $I_c \cap I_a \neq \emptyset$ and hence $I_c \subset I_a$. Since c and d are adjacent, $I_c \cap I_d \neq \emptyset$, hence $I_d \cap I_a \neq \emptyset$ and hence $I_d \subset I_a$. Now if I_b is contained in either I_c or I_d, we see that f_b cannot intersect f_a. Thus, I_b is disjoint from I_c and I_d. Moreover since $l(c) < l(b)$ in the case under consideration, we see that I_b has to be to the right of the interval $I_c \cup I_d$. Hence, $\{c, d\} \prec_r \{a, b\}$ in this case. □

Since chordal graphs are interval filament graphs they belong to ISD(2). Hence, a chordal graph G has induced separation dimension either 0, 1 or 2. It is clear that checking whether $\text{isd}(G) = 0$ can be done in polynomial time. A naive algorithm which tests every triple of edges in G for being an ATE can determine ATE-freeness in poly-time. Hence, by Theorem 12, we can test in poly-time whether $\text{isd}(G) = 1$. In short, we have the following corollary.

Corollary 16. *The induced separation dimension of chordal graphs can be determined in polynomial time.*

5 Graphs with large induced separation dimension

The separation dimension of an n-vertex graph is at most $O(\lg n)$ [3]. Since induced separation dimension of a graph is at most its separation dimension, we observe that the induced separation dimension of an n-vertex graph is at most $O(\lg n)$. In this section, we construct graphs which show that this upper bound is tight up to a constant factor.

Definition 17 (Bipartite cover). Given a graph G, the bipartite cover B_G of G is the direct product of G with K_2. That is, if $V(G) = [n]$, then the two color classes in $V(B_G)$ are $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ with a_i adjacent to b_j in B_G if and only if i is adjacent to j in G.

Theorem 18. *For every graph G,*

$$\text{isd}(B_G) \geq \lg \chi(G),$$

where $\chi(G)$ is the chromatic number of G.

Proof. A linear order $\prec$ of $V(B_G)$ is said to *cover* an edge ij of G if the two strongly independent edges $\{a_i b_j, a_j b_i\}$ are separated in $\prec$. The set of edges of G covered by $\prec$ forms a subgraph of G which we denote by $G_\prec$. We now argue that $G_\prec$ is bipartite for any linear order $\prec$. Color a vertex $i \in V(G)$ white if $a_i \prec b_i$ and black otherwise. If an edge ij belongs to $G_\prec$ then $a_i b_j$ and $a_j b_i$ are separated in $\prec$. This happens only if $a_i \prec b_i$ and $a_j \succ b_j$ or vice versa. In both cases i and j are of different color. Hence, we conclude that $G_\prec$ is a bipartite subgraph of G.

Let $\mathcal{L}$ be a family of total orders which separates every pair of strongly independent edges in B_G. For every edge ij in G, the pair of edges $\{a_i b_j, a_j b_i\}$

are strongly independent in B_G. Hence, every edge of G is covered by at least one linear order in $\mathcal{L}$. It is easy to see that at least $\lg \chi$ bipartite graphs are needed to cover all the edges of a χ-chromatic graph. Hence $|\mathcal{L}| \geq \lg \chi(G)$. □

The bipartite cover of a complete graph is called a *crown graph.* By Theorem 18, we see that the crown graph on $2n$ vertices has induced separation dimension at least $\lg n$. Thus, in general, bipartite graphs have unbounded induced separation dimension.

Another intriguing family is that of AT-free graphs. Since AT-free graphs have a kind of linear structure (dominating pairs) it is tempting to think that their induced separation dimension is at most 1. But we know it is not. The circular ladder CL_k is the graph obtained by taking the Cartesian product of the cycle C_k on $k \geq 3$ vertices with an edge. Orienting a single edge of CL_k forces the orientation on every other edge if we want the orientation to be transitive on each induced C_4. It is easy to check that $\eta(CL_k) \leq 1$ if and only if k is even. Corollary 4 shows that $\mathrm{isd}(\overline{CL_k}) \leq 1$ only when k is even. Notice that for every odd $k \geq 5$, $\overline{CL_k}$ is AT-free (since CL_k is triangle-free) and has induced separation dimension more than 1. Now we amplify this result to show that the induced separation dimension of the family of AT-free graphs is unbounded.

Definition 19 (Double). Given a graph G, the *double* D_G of G is the Cartesian product of G with K_2. That is, D_G consists of two copies of G and a perfect matching of edges between corresponding vertices in the two copies.

Theorem 20. *For every graph G,*

$$\eta(D_G) \geq \lg \chi(G),$$

where $\chi(G)$ is the chromatic number of G.

Proof. To every edge e of G, we associate the induced 4-cycle D_e in D_G formed by the two copies of e and the two matching edges between their endpoints. An acyclic orientation O of D_G is said to *cover* an edge e of G if the associated 4-cycle D_e is oriented transitively by O. The set of edges of G covered by O forms a subgraph of G which we denote by G_O. If G_O contains an odd cycle Z, then it means that O transitively oriented every induced C_4 in the odd circular ladder $D_Z \subset D_G$ which we have observed is impossible. Thus, G_O is bipartite for any acyclic orientation O of D_G.

Let $\mathcal{O}$ be a family of acyclic orientations of D_G such that every induced C_4 in D_G is transitively oriented in at least one orientation in $\mathcal{O}$. Therefore, every edge of G is covered by at least one orientation in $\mathcal{O}$. Hence $|\mathcal{O}| \geq \lg \chi(G)$. □

If G is triangle free, so is D_G and therefore the maximum size of an independent set in $\overline{D_G}$ is 2 and, in particular, $\overline{D_G}$ is AT-free. There are many classic constructions of families of triangle-free graphs with unbounded chromatic number, Mycielski graphs [17] for instance. If $\mathcal{G}$ is a family of triangle-free graphs with unbounded chromatic number, $\left\{\overline{D_G} : G \in \mathcal{G}\right\}$ is a family of AT-free graphs with unbounded induced separation dimension.

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