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Track number of line graphs

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The *track number* $\tau(G)$ of a graph G is the minimum number of interval graphs whose union is G. We show that the track number of the line graph $L(G)$ of a triangle-free graph G is at least $\lg\lg\chi(G)+1$, where $\chi(G)$ is the chromatic number of G. Using this lower bound and two classical Ramsey-theoretic results from literature, we answer two questions posed by Milans, Stolee, and West [J. Combinatorics, 2015] (MSW15). First we show that the track number $\tau(L(K_n))$ of the line graph of the complete graphs K_n is at least $\lg\lg n - o(1)$. This is asymptotically tight and it improves the bound of $\Omega(\lg\lg n/\lg\lg\lg n)$ in MSW15. Next we show that for a family of graphs $\mathcal{G}$, $\{\tau(L(G)) : G \in \mathcal{G}\}$ is bounded if and only if $\{\chi(G) : G \in \mathcal{G}\}$ is bounded. This affirms a conjecture in MSW15. All our lower bounds apply even if one enlarges the covering family from the family of interval graphs to the family of chordal graphs.

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1. Introduction

The *track number* $\tau(G)$ of a graph G is the minimum number of interval graphs whose union is G. Heldt, Knauer, and Ueckerdt [HKU11] conjectured that the track number of line graphs is unbounded. Milans, Stolee, and West [MSW15] proved this conjecture by showing that the track number $\tau(L(K_n))$ of the line graph of the n-vertex complete graph K_n is $\Omega(\lg\lg n/\lg\lg\lg n)$. They suspected that the denominator in the lower bound could be eliminated and also proposed

Conjecture 1.1 (Milans, Stolee, West [MSW15])**.** *For a sequence $(G_n)_{n=1}^{\infty}$ of graphs, if $\chi(G_n)\to\infty$, then $\tau(L(G_n))\to\infty$, where $\chi(G)$ and $L(G)$ denote, respectively, the chromatic number and the line graph of the graph G.*

In this note, we show that $\tau(L(K_n)) = (1 + o(1)) \lg \lg n$ and prove the above conjecture. Milans et al. obtain bounds on $\tau(L(K_n))$ by connecting the problem with two problems in Ramsey theory of ordered hypergraphs. We use results and techniques from a paper by Esperet, Gimbel, and King [EGK10] who studied the covering of line graphs with equivalence relations. The techniques there are close in spirit to that of the Erdös-Szekeres theorem on total orders and hence also Ramsey theoretic. Incidentally, the result of Esperet et al. disproved a conjecture of McClain [McC09] that the line graph of any triangle-free graph can be covered by three equivalence graphs. We first work with triangle-free graphs and then lift the lower bounds obtained there to complete graphs and general graphs using two classical results from Ramsey theory of graphs.

1.1. Notation and preliminaries

All graphs considered in this note are finite, simple and do not contain self-loops. Logarithm to the bases 2 and e are denoted by lg and ln respectively. The line graph $L(G)$ of a graph G is the intersection graph of the edge-set of G. That is, two vertices of $L(G)$ are adjacent in $L(G)$ if and only if the corresponding two edges of G share a common vertex. The chromatic number of a graph G is denoted by $\chi(G)$. The subgraph of a graph G induced on a subset S of the vertices of G is denoted by $G[S]$.

A *chordal graph* is a graph with no induced cycles of length more than three. A graph is an *interval graph* if it can be represented as the intersection graph of intervals on a straight line. An *equivalence graph* is a disjoint union of cliques. The complete graph on n vertices is denoted by K_n.

The *covering number* of a graph G with respect to a family $\mathcal{F}$ of graphs is the minimum number of graphs from $\mathcal{F}$ whose union is G. For example, the *arboricity* $a(G)$, the *equivalence covering number* $\mathrm{eq}(G)$ and the *track number* $\tau(G)$ of a graph G are its covering numbers with respect to the families of forests, equivalence graphs and interval graphs respectively. Equivalence covering number was introduced by Duchet in 1979 [Duc79] and track number was introduced by Gyárfás and West in 1995 [GW95]. For this article, we find it more natural to analyse the covering number with respect to the family of chordal graphs.

Definition 1.2. The *chordal covering number* $\mathrm{cc}(G)$ of a graph G is the minimum number of chordal graphs whose union is G.

Since equivalence graphs are interval graphs, and interval graphs are chordal, every graph G satisfies the inequalities

$$\mathrm{cc}(G) \leq \tau(G) \leq \mathrm{eq}(G). \tag{1}$$

In the course of this note, it will be clear that these parameters are all within a factor of 2 for line graphs of triangle-free graphs. For general graphs, these parameters can be very different. The equivalence covering number of the n-vertex star graph, which is an interval graph, is $n-1$. As far as we have tried, we could not come up with an explicit example of a chordal graph with a large track number. Nevertheless we can use a counting argument to show that the track number of chordal graphs is unbounded. Since an n-vertex interval graph is completely determined by the relative order of the $2n$ endpoints of the intervals in an interval representation, the number of labelled interval graphs on n vertices is at most $(2n)!$. Hence, for any $k \geq 1$, the number of labelled n-vertex graphs which can be written as the union of k interval graphs is at most $\binom{(2n)!}{k}$ which is $2^{O(kn \lg n)}$. On the other hand, the number of labelled n-vertex chordal graphs is at least $2^{\Omega(n^2)}$. One can see this by counting the number of labelled split graphs on n vertices where the first $\left\lfloor \frac{1}{2}n \right\rfloor$ vertices form a clique and the remaining $\left\lceil \frac{1}{2}n \right\rceil$ vertices can pick any subset of the first $\left\lfloor \frac{1}{2}n \right\rfloor$ vertices as its neighbourhood. This shows that the equivalence covering number cannot be bounded above by any function of the track number alone and the track number cannot be bounded above by any function of the chordal covering number alone.

1.2. Background

As mentioned earlier, we use results and techniques from [EGK10] to estimate the chordal covering number of line graphs. The starting point there is a connection that they establish between equivalence coverings of $L(G)$ and a certain family of orientations of G. An *orientation* of an undirected simple graph G is the directed graph formed by assigning one of the two possible orientations to each edge of G. Two incident edges xy and xz of G are said to form an *elbow* in an orientation of G if both of them are directed towards x or if both of them are directed away from x. In the first case, we will call the elbow an *in-elbow* and in the second case, we will call it an *out-elbow*. A family $\mathcal{O}$ of orientations of G such that every pair of incident edges xy and xz in G form an in-elbow (resp., elbow) in at least one of the orientations in $\mathcal{O}$ is called an *in-elbow cover* (resp., *elbow cover*) of G. The minimum size of an in-elbow cover (resp., elbow cover) is denoted by in-elb(G) (resp., elb(G)).

Esperet et al. observed that given an in-elbow cover $\mathcal{O}$ of a graph G, one can construct an equivalence cover of $L(G)$ using $|\mathcal{O}|$ equivalence graphs. The set of vertices forming the j-th clique in the i-th equivalence graph in

the cover of $L(G)$, $1 \leq j \leq |G|$, $1 \leq i \leq |\mathcal{O}|$, is the set of edges incident to and directed towards the j-th vertex of G in the i-th orientation in $\mathcal{O}$. In the other direction, they showed that, given an equivalence cover $\mathcal{F}$ of $L(G)$, one can construct an in-elbow cover of G using $3|\mathcal{F}|$ orientations of G. Let H be an equivalence subgraph of $L(G)$. Every clique in H corresponds to either a set of edges in G containing a common vertex (star-clique) or three edges forming a triangle in G (triangle-clique). Consider the following three orientations of G based on H. The edges of G which form a star-clique in H are oriented towards the common vertex in all the three orientations. The edges of G which form a triangle-clique in H are oriented such that each pair among these three edges form an in-elbow in one of the three orientations. Repeating this for every equivalence graph in an equivalence cover of $L(G)$, they concluded that

$$\frac{1}{3}\,\text{in-elb}(G) \leq \text{eq}(L(G)) \leq \text{in-elb}(G). \tag{2}$$

Similarly, since the three pairs of incident edges in a triangle-clique can be elbow-covered using two orientations, one can also see that

$$\frac{1}{2}\,\text{elb}(G) \leq \text{eq}(L(G)) \leq 2\,\text{elb}(G), \tag{3}$$

where the second inequality follows from the trivial fact that $\text{in-elb}(G) \leq 2\,\text{elb}(G)$.

The first result in this paper is that $\text{elb}(G) \leq cc(L(G))$ when G is triangle-free (Theorem 2.1). Before getting to it, we state and briefly discuss the quantitative connection between the elbow covering number and the chromatic number of a graph that was established in [EGK10].

Theorem 1.3 (Theorem 10 in [EGK10])**.** *For any graph with at least one edge,*

$$\text{elb}(G) = \lceil \lg \lg \chi(G) \rceil + 1.$$

From Theorem 1.3 and the inequalities in (3), it follows that

$$\frac{1}{2}\left(\lceil \lg \lg \chi(G) \rceil + 1\right) \leq \text{eq}(L(G)) \leq 2\left(\lceil \lg \lg \chi(G) \rceil + 1\right). \tag{4}$$

They remarked towards the end of the paper that, using the notion of 3-suitability, one can improve the upper bound to $\lg \lg \chi(G) + \left(\frac{1}{2} + o(1)\right) \lg \lg \lg \chi(G)$. A family $\mathcal{F}$ of total orders of $[n]$ is 3-*suitable* if, for every 3 distinct elements $a, b, c \in [n]$ there exists a total order $\sigma \in \mathcal{F}$ such that a succeeds both b

and c in σ [Dus50]. Following Spencer [Spe72], let $N(n,3)$ denote the cardinality of a smallest family of total orders that is 3-suitable for $[n]$. Very tight estimates which can determine the exact value of $N(n,3)$ for almost all n were given by Hoşten and Morris in 1999 by finding a nice equivalence of this problem to a variant of the Dedekind problem [HM99]. It follows from there that $f(n) - o(1) \leq N(n,3) \leq f(n) + 1 + o(1)$, where $f(n) = \lg\lg n + \frac{1}{2}\lg\lg\lg n + \frac{1}{2}\lg\pi$.

Let $c : V(G) \to [k]$ be a proper vertex colouring of an undirected graph G and let $\mathcal{F}$ be a family of 3-suitable total orders of the colours $[k]$. For each total order in $\sigma \in \mathcal{F}$ construct an orientation of G by directing each edge xy from x to y if $c(x)$ precedes $c(y)$ in σ and the opposite otherwise. It is easy to verify that this family of orientations is an in-elbow cover of G. Hence in-elb(G) and thereby eq$(L(G))$ is at most $N(\chi(G),3)$.

2. Chordal covering number of line graphs

In this section we first show that, for a triangle-free graph G, the chordal covering number of $L(G)$ is at least the elbow covering number of G. This lower bound can be written in terms of $\chi(G)$ using Theorem 1.3. Using this lower bound and two classical Ramsey-theoretic results from literature, we answer two questions posed by Milans, Stolee, and West [MSW15].

A *simplicial vertex* in a graph G is one whose neighbourhood induces a clique in G. A *perfect elimination ordering* of G is an ordering $(v_1, \ldots, v_n)$ of $V(G)$ such that v_i is a simplicial vertex in $G[\{v_i, \ldots, v_n\}]$, for each i. It is well known that a graph has a perfect elimination ordering if and only if it is chordal [FG65].

Theorem 2.1. *For every triangle-free graph G,*

$$\text{elb}(G) \leq \text{cc}(L(G)).$$

Proof. Let G be any triangle-free graph. Let $\mathcal{H}$ be a smallest collection of chordal graphs whose union is $L(G)$. Based on each chordal graph $H \in \mathcal{H}$, we construct an orientation O_H of G such that every pair of incident edges of G which are adjacent as vertices in H will form an elbow in O_H. Since every pair of incident edges of G are adjacent as vertices in at least one H in $\mathcal{H}$, it is easy to verify that the family of $|\mathcal{H}|$ orientations constructed with the promised property will serve as an elbow-cover of G with size cc$(L(G))$.

Let $H \in \mathcal{H}$ be arbitrary. By allowing isolated vertices if necessary, we assume that H is a spanning subgraph of $L(G)$. Consider a perfect elimination ordering $e_1, \ldots, e_m$ of H, where m is the number of edges in G. That is,

$\forall i \in [m]$, e_i is a simplicial vertex in $H_i = H[\{e_i, \ldots, e_m\}]$. For each i going from m down to 1, the edge e_i in G is oriented so that it forms an elbow in O_H with the most recently oriented edge of G which is adjacent as a vertex to e_i in H_i. If e_i has no neighbours in H_i, then it is oriented arbitrarily.

Now we argue that every pair of incident edges in G which are adjacent as vertices in H will be oriented to form an elbow in O_H. For each $i \in [m]$, let N_i denote the neighbours of e_i in H_i. We call an edge e_i in O_H "good" if it forms an elbow in O_H with every edge of G which belongs to N_i. It is enough to show that every edge $e_i, i \in [m]$ is good. We show this by induction on $(m-i)$. Vacuously, e_m is good. For some $i < m$, let us assume, by induction, that $e_{i'}$ is good for all $i' > i$. If $|N_i| \leq 1$, then it is clear that e_i will be good. If $|N_i| \geq 2$, let $j = \min\{k : e_k \in N_i\}$. By construction e_i and e_j form an elbow in O_H. Moreover, e_j is good in H_j and hence e_j forms an elbow with every edge in N_j. Since e_i is simplicial, $N_i \cup \{e_i\}$ induces a clique in H_i; that is, these edges are pairwise incident in G. Since G is triangle-free, these edges share a common vertex. Since e_i forms an elbow with e_j and e_j forms an elbow with every edge in N_j, which is a superset of $N_i \setminus \{e_j\}$, □

Remark. From Theorem 2.1, (1) and (3), we see that for every triangle-free graph G,

$$\mathrm{cc}(L(G)) \leq \tau(L(G)) \leq \mathrm{eq}(L(G)) \leq 2\,\mathrm{elb}(L(G)) \leq 2\,\mathrm{cc}(L(G)).$$

From Theorem 1.3 and Theorem 2.1 one can immediately infer

Corollary 2.2. *For every triangle-free graph G,*

$$\lceil \lg \lg \chi(G) \rceil + 1 \leq \mathrm{cc}(L(G)).$$

We can use the above result together with some celebrated Ramsey-theoretic results to estimate the chordal covering number of complete graphs and general graphs. Since the family of chordal graphs is hereditary, $\mathrm{cc}(G') \leq \mathrm{cc}(G)$ whenever G' is an induced subgraph of a graph G. Since the line graph of a subgraph is an induced subgraph of the line graph of the original graph, $\mathrm{cc}(L(H')) \leq \mathrm{cc}(L(H))$ whenever H' is a subgraph of H.

It was established by Kim [Kim95] that for every sufficiently large n, there exists an n-vertex triangle-free graph G_n with

$$\chi(G_n) \geq \frac{1}{9}\sqrt{\frac{n}{\ln n}}.$$

Since G_n is a subgraph of K_n, $\mathrm{cc}(L(G_n)) \leq \mathrm{cc}(L(K_n))$. This gives the lower bound in

Corollary 2.3.

$$\lg\lg n - o(1) \leq \mathrm{cc}(L(K_n)) \leq \tau(L(K_n)) \leq \lg\lg n + \frac{1}{2}\lg\lg\lg n + \frac{1}{2}\lg\pi + 1 + o(1).$$

The upper bound follows from the inequality $\mathrm{eq}(L(K_n)) \leq N(n,3)$. So we can remove the denominator from the lower bound of $\Omega(\lg\lg n/\lg\lg\lg n)$ on $\tau(L(K_n))$ from [MSW15] as suspected by the authors. Furthermore, it shows that $\tau(L(K_n))$ is asymptotically $(1+o(1))\lg\lg n$.

Finally, we use these two results together with a beautiful result of Rödl to prove Conjecture 1.1. It was shown by Rödl [Röd77] that, for arbitrary positive integers m and n, there exits a $\phi(m,n)$ such that if $\chi(G) \geq \phi(m,n)$, then the graph G contains either a clique of size m or a triangle-free subgraph H with $\chi(H) = n$. Consider any sequence $(G_n)_{n=1}^{\infty}$ of graphs, with $\chi(G_n) \to \infty$. Suppose $\tau(L(G_n))$ was bounded above by some constant b. Let $B = 2^{2^{b+1}}$ and choose a graph G from the sequence (G_n) with $\chi(G) \geq \phi(B,B)$. Using Rödl's result, we can conclude that G either contains a B-vertex complete graph K_B or a triangle-free graph H with $\chi(H) = B$. In either case, we have shown that the chordal covering number of the line graph of that subgraph is more than b (Corollaries 2.3 and 2.2). This contradiction proves

Theorem 2.4. *For a sequence $(G_n)_{n=1}^{\infty}$ of graphs, if $\chi(G_n) \to \infty$, then $\mathrm{cc}(L(G_n)) \to \infty$.*

Thus we affirm Conjecture 1.1. Further, since $\mathrm{cc}(L(G)) \leq \tau(L(G)) \leq \mathrm{eq}(L(G)) \leq N(\chi(G),3)$, we see that, for a family of graphs $\mathcal{G}$, $\{\tau(L(G)) : G \in \mathcal{G}\}$ is bounded if and only if $\{\chi(G) : G \in \mathcal{G}\}$ is bounded.

3. Concluding remarks

The function $\phi(m,n)$ obtained by Rödl is a tower of n's of height m. Hence the lower bound obtained for $\tau(L(G))$ for a general graph G in terms of $\chi(G)$ is of very small order. We suspect that, like $\mathrm{eq}(L(G))$, $\mathrm{cc}(L(G))$ might also be bounded below by $\Omega(\lg\lg\chi(G))$.

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